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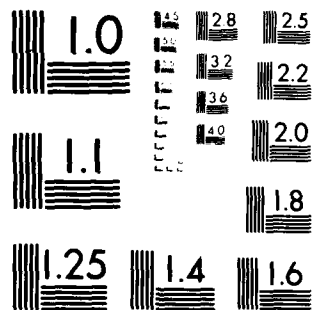
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AN ANALYSIS OF THE VARIOUS METHODS USED TO ANALYZE RADIATION FROM THE THIN WIRE E-FIELD INTEGRAL EQUATION

Tapan K. Sarkar

ABSTRACT

In this paper we analyze the numerical aspects of the various methods that have been used to analyze thin wire antenna problems. First we derive some properties of the thin wire E-field integral operator. Based on those properties we unify the various iterative methods used to find current distribution on thin wire structures. An attempt has been made to resolve the question of numerical stability associated with various entire domain and subdomain expansion functions in Galerkin's method. It has been shown that the sequence of solutions generated by the iterative methods monotonically approaches the exact solution provided the excitation chosen for these problems are in the range of E-field operator. Such a statement does not hold for Galerkin's method since the inverse operator is unbounded. Moreover if the excitation function is not in the range of the operator the sequence of solutions form an asymptotic series. Examples have been presented to illustrate this point.

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I. INTRODUCTION

Over the past thirty years several methods have been developed by various researchers to analyze scattering and radiation from thin wire structures. In this presentation we investigate the properties of the integrodifferential equations by various techniques which impose boundary conditions on the electric field only. So we restrict our discussions to the E-field integral equation.

For simplicity let us focus our attention on the reradiation of electromagnetic waves by a thin wire of radius a and of length L centered at $z = 0$. In this analysis we will assume that the antenna is quite thin and hence the circumferential variation of the current on the antenna can be neglected. We shall also neglect any ϕ component of the current on the structure. Furthermore, there are no internal resonances of the thin wire at the frequency of operation. With these assumptions we derive the properties of the E-field operator for both the Pocklington and Hallen's equations.

2. PROPERTY OF THE POCKLINGTON E-FIELD OPERATOR

By assuming a time variation of the form $\exp(j\omega t)$ the Pocklington integral equation for the current on the surface of antenna can be written as [1]

$$k^2 \int_{-L/2}^{+L/2} dz' \int_0^{2\pi} d\phi I(z') G(z, z') + \frac{\partial^2}{\partial z^2} \int_{-L/2}^{+L/2} dz' \int_0^{2\pi} d\phi I(z') G(z, z') = -j\omega 4\pi \epsilon E^i(z)$$

for $-\frac{L}{2} < z < +\frac{L}{2}$ (1)

$$\text{where } G(z, z') = \frac{1}{2\pi} \frac{\exp(-jkR)}{R} \quad (2)$$

$$R = \sqrt{(z-z')^2 + 4a^2 \sin^2 \frac{\phi}{2}} \quad (3)$$

$$E^i(z) = \text{incident field on the antenna} \quad (4)$$

In the operator form (1) can be written as

$$PI \triangleq k^2 P_1 I + P_2 I = V \quad (5)$$

We next investigate if there exists a constant C independent of I such that [2, p.296]

$$\|P\| = \max_{\|I\|=1} \frac{\|PI\|}{\|I\|} = \max_{\|I\|=1} \|PI\| \leq C \quad (6)$$

If such a constant C exists which is the maximum of all possible $\|PI\|$ with the constraint $\|I\| = 1$ then we say the operator is bounded with respect to that norm $\|\cdot\|$. The norms that we shall be dealing with are the L^2 norm and the Chebyshev norm. The L^2 norm is defined as

$$\|I\|_2 = \left[\int_{-L/2}^{+L/2} |I(z)|^2 dz \right]^{1/2} \quad (7)$$

and the Chebyshev norm is defined as

$$\|I\|_T = \max_{-\frac{L}{2} \leq z \leq +\frac{L}{2}} |I(z)| \quad (8)$$

When no subscripts are used either norm is implied. If we are using the \mathcal{L}^2 norm then we are restricting the domain of the operator P to elements which are in \mathcal{L}^2 (or square integrable). This does not imply that $I(z')$ cannot be infinite within the range $-\frac{L}{2} \leq z' \leq +\frac{L}{2}$. However only those type of singularities are permitted in $I(z')$ which are square integrable. Any function which is not square integrable is excluded from the domain of P (as they are not in \mathcal{L}^2). On the other hand, if we use the Chebyshev norm then the function has to be bounded. Under the Chebyshev norm any unbounded function cannot be in the domain of the operator. Thus the function $\log z$ is in \mathcal{L}^2 but not in the domain of functions bounded under the Chebyshev norm. Physically then convergence of a sequence of functions under the \mathcal{L}^2 norm yields least squares convergence whereas convergence under the Chebyshev norm yields pointwise convergence.

Examination of the first part of the integral in (1) reveals that the kernel has a singularity which can be observed by rewriting the kernel as [3, p. 141]

$$\int_0^{2\pi} d\phi G(z, z') = \frac{1}{2\pi} \int_0^{2\pi} \frac{\exp(-jkR)}{R} d\phi = \log |z - z'| + G_2 \quad (9)$$

where G_2 contains terms which are bounded and hence square integrable. The singularity of the kernel in P_1 is manifested through the function $\log |z - z'|$. Since a logarithmic function is square integrable we find

$$\|P_1\|_{\mathcal{L}^2} \leq \left[\int_{-L/2}^{+L/2} dz \int_{-L/2}^{+L/2} dz' \{ \log |z-z'| + G_2 \}^2 \right]^{1/2} = C < \infty \quad (10)$$

where C is a constant. Hence P_1 is bounded under the \mathcal{L}^2 norm. Furthermore P_1 is a Hilbert Schmidt operator as it has a square integrable kernel [2, p. 352]. It can also be shown that a Hilbert Schmidt operator is a compact operator [2, p. 353]. Under the Chebyshev norm

$$\begin{aligned} \|P_1\|_T &\leq \max_{-\frac{L}{2} \leq z \leq +\frac{L}{2}} \left| \int_{-L/2}^{+L/2} dz' \log |z-z'| \right| + \max_{-\frac{L}{2} \leq z \leq +\frac{L}{2}} \int_{-L/2}^{+L/2} dz' |G_2| \\ &= \max_{-\frac{L}{2} < z < \frac{L}{2}} \left| \left(z + \frac{L}{2} \right) \log \left(z + \frac{L}{2} \right) + \left(\frac{L}{2} - z \right) \log \left(\frac{L}{2} - z \right) - L \right| + \text{constant} \\ &\leq M \text{ (a constant)} \quad \text{for all } -\frac{L}{2} \leq z \leq +\frac{L}{2} \quad (11) \end{aligned}$$

Hence the operator P_1 is also bounded under the Chebyshev norm.

Next consider the second integral in (1). We have

$$P_2 I = \frac{\partial}{\partial z} \int_{-L/2}^{+L/2} dz' \frac{\partial \bar{I}}{\partial z}, \{ \log |z-z'| + G_2 \} = \int_{-L/2}^{+L/2} dz' \frac{\partial \bar{I}}{\partial z}, \frac{\partial}{\partial z} \{ \log |z-z'| + G_2 \} \quad \dots (12)$$

where the bar over the second integral represents a principal value. It is clear that the operator P_2 in (12) is unbounded under the Chebyshev norm, because $\frac{\partial \bar{I}}{\partial z}$ is unbounded as $z \rightarrow \pm \frac{L}{2}$. Thus there exist no constant C for all $-\frac{L}{2} < z < +\frac{L}{2}$ such that

$$\|P_2\|_T \leq C < \infty$$

Also the operator P_2 is unbounded under the \mathcal{L}^2 norm as $\frac{\partial I}{\partial z'}$, and $\frac{\partial}{\partial z} \{ \log |z - z'| + G_2 \}$ are not square integrable.

However if the antenna has no edges (i.e. $\frac{\partial I}{\partial z}$ is everywhere bounded and square integrable) then it can be shown that the operator P_2 is bounded under the \mathcal{L}^2 norm. Sneddon has shown through Theorem 8 [10, p. 234] that if $f(z)$ is square integrable over $-\frac{L}{2} \leq z \leq \frac{L}{2}$ and zero everywhere else then the formula $\bar{f}_H(z) = -\frac{1}{\pi} \frac{d}{dz} \int_{-\infty}^{\infty} f(t) \log \frac{|t-z|}{|t|} dt$ defines almost everywhere a function $\bar{f}_H(z)$ which is also square integrable, and

$$\|f(z)\|_{\mathcal{L}^2} = \|\bar{f}_H(z)\|_{\mathcal{L}^2}$$

Hence if the antenna has no sharp edges then the operator P_2 is bounded under the \mathcal{L}^2 norm as illustrated by the above theorem. In other words, if the antenna has end caps then the Pocklington E-field operator is bounded.

Hence the Pocklington E-field operator, $P = P_1 + P_2$ is unbounded if the scatterer has a sharp edge at which the charge density becomes infinite. However if the antenna structure is quite smooth, so that at each point the tangent to the surface can be defined uniquely, then the Pocklington E-field operator $P = P_1 + P_2$ is bounded under the \mathcal{L}^2 norm.

Next we consider the Hallen E-field operator.

3. PROPERTY OF THE HALLEN E-FIELD OPERATOR

Hallen transformed Pocklington's equation as given by (1) into the following integral equation

$$\int_{-L/2}^{+L/2} dz' \int_0^{2\pi} d\phi I(z') G(z, z') = D \cos kz + F \sin kz +$$

$$\frac{jk}{\omega} \int_{-L/2}^z E^i(z) \sin k(z - z') dz' \quad (14)$$

where D and F are constants which depend on how the current goes to zero at the ends of the antenna. We define the Hallen E-field operator as

$$HI = \int_{-L/2}^{+L/2} dz' \int_0^{2\pi} d\phi I(z') G(z, z') = P_1 I \text{ [from (1)]} \quad (15)$$

Hence the operator H is bounded both under the \mathcal{L}^2 and the Chebyshev norm. Also H is a compact operator under the \mathcal{L}^2 norm.

It is important to note however, that the unknown $I(z')$ is hidden in the constants D and F. To illustrate this further, if we consider a delta gap excitation for the antenna then (14) becomes [4, p. 325]

$$\int_{-L/2}^{+L/2} dz' \int_0^{2\pi} d\phi I(z') G(z, z') = A \sin k |z| + D \cos kz$$

where A is a known constant and D is unknown. Observe at $z = 0$

$$D = \int_{-L/2}^{+L/2} dz' \int_0^{2\pi} d\phi I(z') G(0, z')$$

If the operator H is bounded so will be the constant D. If one wishes then perhaps one can transfer D to the left hand side of the equation and thus form an additional part of the operator H. But since D is a part of the operator H, whatever bounds hold for H also hold for D.

Next we estimate a bound for $\|H\|$ both under the \mathcal{L}^2 and Chebyshev norm.

We observe

$$\|H\|_T \leq \max_{-\frac{L}{2} \leq z \leq +\frac{L}{2}} \frac{1}{2\pi} \int_{-L/2}^{+L/2} dz' \int_0^{2\pi} d\phi \frac{\exp(-jkR)}{R}$$

$$\leq \max_{-\frac{L}{2} \leq z \leq +\frac{L}{2}} \frac{1}{2\pi} \int_{-L/2}^{+L/2} dz' \int_0^{2\pi} d\phi \frac{1}{R}$$

$$= \max \left[2 \log \frac{L}{a}, \log \frac{2L}{a} \right] + \theta(a^2) \cdot [3, p. 144] \quad (16)$$

where $\theta(a^2)$ denotes terms of the order of a^2 . Now if $\frac{L}{a} \gg 1$ then

$$||H||_T \leq 2 \log \frac{L}{a} \quad (17)$$

Under the \mathcal{L}^2 norm

$$||H||_{\mathcal{L}^2} \leq \left[\int_{-L/2}^{+L/2} dz \int_{-L/2}^{+L/2} dz' \left\{ \int_0^{2\pi} d\phi \frac{\exp(-jkR)}{2\pi R} \right\}^2 \right]^{1/2}$$

$$\leq \left[\int_{-L/2}^{+L/2} dz \int_{-L/2}^{+L/2} dz' \left\{ \int_0^{2\pi} d\phi \frac{1}{2\pi R} \right\}^2 \right]^{1/2}$$

$$\leq \theta \left[\frac{L}{a} \log \frac{L}{a} \right] + \text{other terms} \quad (18)$$

The above integrals may be evaluated numerically to give a more accurate estimate for $||H||_{\mathcal{L}^2}$.

As H is bounded operator unlike P , it may be computationally much easier to solve Hallen integral equation than Pocklington integral equation.

4. SOLUTION OF HALLEN E-FIELD INTEGRAL EQUATION

4.1. By Iterative Methods

It is well known that if H is a compact invertible operator (under \mathcal{L}^2 norm) on an infinite dimensional space then its inverse is often unbounded [3, p. 353]. Hence the problem of the solution of (14) in the

\mathcal{L}^2 norm is ill-posed. If a problem is ill-posed under the \mathcal{L}^2 norm then it is definitely ill-posed under the Chebyshev norm. However it can be regularized in the following way. We take (14) and cast it in the form

$$HI = Q \quad (19)$$

and generate the sequence $\{I_n\}$

$$I_n = [U - \tau H] I_{n-1} + \tau Q \quad (20)$$

with a starting guess of $I_0 = Q$ and U is the identity operator. The sequence I_n generated by (20) converges to a solution I which satisfies $HI = Q$ for all Q in the range of H [1, p. 196]. The sequence generated by (20) always converge to I provided

$$\begin{aligned} & || [U - \tau H] || < 1 \\ \text{or} \quad & |\tau| \cdot ||H|| = ||\tau H - U + U|| \leq ||U - \tau H|| + ||U|| < 2 \\ \text{or} \quad & \frac{1}{|\tau|} > \frac{||H||}{2} \end{aligned} \quad (21)$$

In (21) $||H||$ could be either the \mathcal{L}^2 or the Chebyshev norm depending on the type of convergence desired. For all values of $\frac{1}{|\tau|} > \frac{||H||}{2}$ and Q in the range of the operator H , the iterative process defined by (20) will always converge monotonically to a solution $I(z')$, if it exists. This has been shown in Theorem 2 (in the appendix). By the terms of Theorems 1 and 2, the iterative process will converge for any starting value I_0 if

$$\frac{1}{|\tau|_{\mathcal{L}^2}} > \frac{||H||_{\mathcal{L}^2}}{2} \quad \text{or} \quad \frac{1}{|\tau|_{\mathcal{L}^2}} \approx \theta\left(\frac{1}{a} \log \frac{L}{a}\right) \quad (22)$$

(when convergence is desired in χ^2 norm); and

$$\frac{1}{|\tau|_T} > \frac{||H||_T}{2} \quad \text{or} \quad \frac{1}{|\tau|_T} \approx 2 \log \frac{L}{a} \quad (23)$$

(when convergence is desired in Chebyshev norm).

Observe if a χ^2 solution is desired and one chooses τ according to (23)

it is quite possible that the iterative method described by (20) may not converge at all. Therefore if we are solving for the current distribution on a thin wire due to a delta excitation the value of τ chosen should be $\theta \left[\frac{L}{a} \log \frac{L}{a} \right]$ instead of the popularly used value of $\tau = 2 \log \frac{L}{a}$. This is because the current has a logarithmic singularity at $z = 0$ and so convergence of the various iterations can be guaranteed only in an χ^2 sense.

Hallen in his classic iterative scheme chose the value of τ as given by (23) [3, 4]. A detailed description on how D is solved for at each iteration is described in detail [4, p. 326]. Observe that if $\frac{1}{\tau} > \frac{||H||}{2}$ or $||U - \tau H|| < 1$ then the iterations defined by {(8.149) and (8.150) of [4]} would always converge for any starting guess I_0 .

Other researchers have chosen different values of τ . For example Gray [5] chose

$$\frac{1}{\tau} = \text{Real} \left[2 \log \frac{L}{a} - 2\gamma - 2 \log \frac{kL}{2} - j\pi + 2\text{Ei} \left(-\frac{j kL}{2} \right) \right] \quad (24)$$

where γ is Euler's constant and Ei is the exponential integral. King and Middleton [6] decided to make

$$\frac{1}{\tau} = \int_0^{L/2} G\left(L - \frac{\lambda}{4}, z'\right) \sin kz' dz'. \quad (24.a)$$

Whereas Siegel and Labus [7] chose

$$\frac{1}{\tau} = 2 \log \frac{L}{a} - \text{Cin}(kL) - 1 - \frac{\sin kL}{kL} \quad (25)$$

Where Cin is the special form of the cosine integral. Finally Schelkunoff [3] after a careful analysis decided

$$\frac{1}{\tau} = 2 \log \frac{L}{a} - \text{Cin}(kL) - 1 - \frac{\text{Sin } kL}{kL} - j \text{Sin}(kL) + j \frac{1 - \cos kL}{kL} \quad (25.a)$$

In general, it really does not make any difference whatsoever, what value of τ one chooses, one is guaranteed to have pointwise convergence or convergence in the mean depending on whether one chooses τ according to (23) or (22). This of course assumes that a solution to the problem exist, i.e. Q is in the range of H .

In summary, the iterative method convert $HI = Q$, a Fredholm equation of the first kind to $I_n = B I_{n-1} + \tau Q$, as Fredholm equation of the second kind. The advantage of the equation of the second kind is that $||[U - B]||$ is not only bounded but its inverse $||[U - B]^{-1}||$ is also bounded, provided unity is not an eigenvalue of B . Mathematically one has regularized the problem by the introduction of the parameter τ . With this regularization scheme the convergence of the sequence I_n is monotonic.

Finally, we conclude by noting that as the iterative process continues the unknown constants D and F in (14) are determined as outlined in [4].

4.2. By Galerkin's Method

The next generation of the methods were developed primarily by Harrington [8] under the generic name of "moment methods." This very popular versatile method has been excellently documented in [8]. In Galerkin's method, the unknown function I is expressed as

$$I_N(z) = \sum_{i=1}^N \alpha_i \psi_i(z) \quad (26)$$

where $\psi_i(z)$ are known functions which may extend from $-\frac{L}{2} \leq z \leq +\frac{L}{2}$ or could span only a partial portion of the domain of z , i.e.

$-\frac{L}{2} < \sigma_1 \leq z \leq \sigma_2 < +\frac{L}{2}$. In the former case ψ_i 's become entire domain functions whereas in the latter ψ_i 's are called sub domain basis functions.

We solve for $I_N(z)$ by solving for the unknowns α_i in (26). We also convert the infinite dimensional problem $HI = Q$ to a finite dimensional problem by replacing I with I_N , i.e. we solve the following equation

$\sum_{i=1}^N \alpha_i H\psi_i = Q$ in the finite dimensional space spanned by the basis functions ψ_i , $i = 1, 2, \dots, N$. We next find a unique solution in finite dimensional space by weighting the residual $\sum_{i=1}^N \alpha_i H\psi_i - Q$ to zero in the following way

$$\sum_{i=1}^N \alpha_i \langle H\psi_i, \psi_j \rangle = \langle Q, \psi_j \rangle \quad \text{for } j = 1, 2, \dots, N \quad (27)$$

In a matrix form

$$[G] [\alpha] = [V] \quad (28)$$

where $[G] = [\langle H\psi_i, \psi_j \rangle]$

$$[V] = [\langle Q, \psi_j \rangle]$$

and the inner product is defined as

$$\langle \phi_i, \phi_j \rangle = \int_{-L/2}^{+L/2} dz \phi_i(z) \phi_j(z)$$

The unknown α 's in (28) are obtained as

$$[\alpha] = [G]^{-1} [V] \quad (29)$$

The next question that normally arises is whether the sequence I_N defined in (26) approaches any limit I as $N \rightarrow \infty$. And secondly whether I satisfies the equation $HI = Q$. We cannot talk about convergence in the Chebyshev metric [as defined in (8)] because a Chebyshev norm cannot be derived from an inner product [2. p. 272]. In other words in an inner product space we cannot define a Chebyshev norm. Hence we shall be talking about only the L^2 norm for Galerkin's method. So we shall be discussing about convergence in the mean. Galerkin's method guarantees the weak convergence of the residuals [from (27)], i.e.

$$\lim_{N \rightarrow \infty} \langle HI_N - Q, \psi_j \rangle \rightarrow 0 \quad \text{for } j = 1, 2, \dots, N \quad (30)$$

However if H is a bounded operator (i.e. $\|H\|_{L^2} \leq \text{a constant} < \infty$) then (30) implies strong convergence of the residuals, i.e.

$$\lim_{N \rightarrow \infty} \|HI_N - Q\|_{L^2} \rightarrow 0 \quad (31)$$

This has been proved by Mikhlin [9]. Physically, (31) implies that as $N \rightarrow \infty$, the tangential electric field on the surface of the conductor converges to zero in a least squares fashion.

Unfortunately in Galerkin's method the convergence of the residuals to zero in (31) does not imply the convergence of I_N to a solution I of $HI = Q$. The convergence of $I_N \rightarrow I$ in the domain of H is possible if and only if $\|H^{-1}\|_{L^2}$ is bounded, as

$$\|I_N - I\|_{L^2} \leq \|H^{-1}\|_{L^2} \cdot \|HI_N - Q\|_{L^2} \quad (32)$$

So if $\|H^{-1}\|_{L^2}$ is unbounded, even though the residuals go to zero, the sequence of solutions I_N may not converge to I . This is in contrast to

the iterative methods where monotonic convergence to I is guaranteed if τ and Q are chosen as prescribed.

Since $\|H^{-1}\|_{\mathcal{L}^2}$ is unbounded in this case, the application of Galerkin's method to $HI = Q$ may not guarantee that $\|I_N - I\|_{\mathcal{L}^2} \rightarrow 0$ as $N \rightarrow \infty$. In other words, there is no quantitative way to describe the convergence of $I_N \rightarrow I$ as various expansion functions are chosen for ψ_i . Hence we address the question: For a fixed order of approximation N , how should one choose a set of expansion functions ψ_i such that the round-off and the truncation error in the numerical computation of α in (29) is a minimum?

Suppose the Gram matrix E is generated by the basis functions $[E_{ij} = \langle \psi_i, \psi_j \rangle]$ then we show in the appendix (Theorem 3) that

$$\text{cond } [G] \leq \text{cond } [\hat{H}]. \text{ cond } [E] \quad (33)$$

i.e. the condition number of the Galerkin matrix G in (28) is bounded by the condition number of the operator H in the finite N dimensional space and the Gram matrix E . Equation (33) is valid only in the finite N dimensional space spanned by ψ_i . It is important to note that even though H may not have any eigenvalues in an infinite dimensional space, it has at least an eigenvalue on a finite dimensional space [2, p. 332]. If the homogeneous equation $HI = 0$ has only the trivial solution $I = 0$ and $\|H\|_{\mathcal{L}^2}$ is bounded then $\text{cond } [\hat{H}] < \infty$ and the inequality in (33) has some meaning because the right hand side of (33) can never be infinity.

So (33) directly implies the following: 1) Use of an orthonormal set of basis functions ψ_i for the current implies

$$\text{cond } [G] \leq \text{cond } [\hat{H}] \quad (34)$$

i.e. the problem would not be worse conditioned as the original problem. For this case $\text{cond } [E] = 1$. This will happen for subdomain basis functions like pulses or entire domain orthonormal basis functions like $\sqrt{\frac{2}{\pi}} \sin m z$ for $m = 1, 2, \dots, N$. (34) also implies that the solution of $HI = Q$ by Galerkin's method in a finite dimensional space may be a better conditioned problem than the original problem posed in the finite dimensional space N . This definitely should be a very strong point for Galerkin's method.

Also from (34) there is no way to tell whether the Galerkin matrix G associated with the entire domain basis functions would be more ill-conditioned than the Galerkin matrix associated with the pulse functions.

ii) Use of subdomain basis functions like triangles or piecewise sinusoids may deteriorate the condition number of the Galerkin matrix $[G]$ than that of the original problem. This is because $\text{cond } [E] > 1$ for these cases.

For the case when ψ_1 's are chosen as piecewise triangles, then E is a tridiagonal matrix of the form

$$\begin{bmatrix} P & Q & 0 & . \\ & Q & P & Q \\ & 0 & Q & P \end{bmatrix}$$

where $P = \frac{2\Delta z}{3}$ and $Q = \frac{\Delta z}{6}$ and $\Delta z = \frac{L}{N+1}$. Since the j th eigenvalue of a tridiagonal matrix is given by [1, p. 70]

$$\lambda_j = P + 2Q \cos \left(\frac{j\pi}{N+1} \right)$$

we have

$$\text{cond } [E]_{\text{triangles}} \leq \frac{|P| + 2|Q|}{|P| - 2|Q|} = 3 \quad (36)$$

Hence for all dimension N the Galerkin matrix due to piecewise triangle expansion functions may have a condition number which at most can be three times as that of the original problem, i.e.

$$\text{cond } [G] \leq 3 \text{ cond } [H]$$

For the piecewise sinusoids however,

$$P = \frac{2k \Delta z - \sin 2k \Delta z}{2k \sin^2 k \Delta z} \quad \text{and} \quad Q = \frac{\sin k \Delta z - k \Delta z \cos k \Delta z}{2k \sin^2 k \Delta z} \quad (37)$$

where $k = \frac{2\pi}{\text{wavelength}}$. In this case $\text{cond } [E]$ is bounded by

$$\text{Cond } [E]_{\text{Sinusoids}} < \frac{|2k \Delta z - \sin 2k \Delta z| + 2 |\sin k \Delta z - k \Delta z \cos k \Delta z|}{|2k \Delta z - \sin 2k \Delta z| - 2 |\sin k \Delta z - k \Delta z \cos k \Delta z|} \dots (38)$$

In the limit $\Delta z \rightarrow 0$

$$\text{cond } [E]_{\text{Sinusoids}} \leq 3 \quad (39)$$

Thus (39) implies that as the dimension of the problem becomes large the Galerkin matrix due to piecewise sinusoids are no less numerically ill-conditioned than the matrix produced by piecewise triangles. It may be quite possible that for a particular value of N the Galerkin matrix due to piecewise sinusoidal functions may be better conditioned than that of the piecewise triangles or even than that of the pulse functions.

In the above analysis an attempt has been made to provide a worst case theoretical bound for the condition number of the various matrices of interest.

It is important to stress that the problem we have addressed here is

not which set of basis functions would provide the best approximation for the current, but which type of expansion functions would give rise to a well conditioned Galerkin matrix G which will be easy to invert numerically. This is because truncation and round-off error associated with the solution of (28) is directly related to $\text{cond } [G]$.

5. IS A SOLUTION POSSIBLE IF THE EXCITATION IS NOT IN THE RANGE OF THE OPERATOR?

We discuss the question of existence of a solution for the current on the antenna structure when we try to excite it with a source which is not in the range of the operator H . Clearly, if the excitation is not in the range of the operator then mathematically a solution does not exist. But numerically one could always find a solution to the integral equation. This numerical solution has some very interesting properties as outlined in Theorems 4 and 5 (in the appendix). If we try to numerically solve an integral equation $HI = Q$ with $Q \notin \text{range of } H$, then the sequence of solutions I_N diverges even though the residuals $HI_N - Q$ associated with $HI = Q$ may approach zero monotonically. This has been proved in Theorem 4 of the appendix. In theorem 5, we develop further properties of the solution I_N . There we prove that the sequence I_N indeed form an asymptotic series. The asymptotic series has the property that it converges at first and then as more and more terms are included in the series, the series actually diverges. Even though the theorems 4 and 5 have been proved for the iterative methods, they are also valid for Galerkin's method. We now present some examples to illustrate when Q is in the range of operator and when it is not.

As a first example consider the radiation problem where an antenna of length L and radius a is excited by a delta gap at the center. The corresponding Hallen's integral equation has the following form [4, p. 321]

$$\int_{-L/2}^{+L/2} dz' \int_0^{2\pi} d\phi I(z') G(z, z') = A \sin k |z| + D \cos k z$$

where A is a known constant and D is unknown. It has been shown by Wu [4, p. 322] that the solution $I(z)$ of the above equation has a logarithmic singularity at $z = 0$ given by

$$I(z) = -j \frac{4k a}{120\pi} V \left[\log \frac{1}{|z|} + \dots \right]$$

Hence $I(z)$ to the delta excitation is square integrable but is not bounded. So the delta function excitation would be in the range of the operator if the L^2 norm is used. The delta function excitation would not be in the range of the operator if the Chebyshev norm is used.

As a second example consider the scattering from a wire of length L and radius a irradiated by a broadside incident plane wave. In this case the integral equation would be of the form

$$\int_{-L/2}^{+L/2} dz' \int_0^{2\pi} d\phi I(z') G(z, z') = D \cos k z + F \sin k z + Y$$

where D , F and Y are constants. By superposition principle there must exist a current $I_Y(z')$ such that

$$\begin{aligned} \int_{-L/2}^{+L/2} dz' \int_0^{2\pi} d\phi I_Y(z') G(z, z') &= \text{a constant} \\ \int_{-L/2}^{+L/2} dz' \int_0^{2\pi} d\phi \frac{dI_Y(z')}{dz'} G(z, z') &= 0 \end{aligned} \quad (40)$$

If we assume that the function $I_Y(z')$ is square integrable then we can expand

$$I_Y(z') = \sum_{k=1}^N \alpha_k P_k(z) \quad (41)$$

where $P_k(z) = 1$ for $z_k < z \leq z_{k+1}$
 $= 0$ otherwise

By substituting (41) into (40) yields

$$\sum_{k=1}^N \alpha_k \int_{-L/2}^{+L/2} dz' \int_0^{2\pi} d\phi [G(z, z_k) - G(z, z_{k+1})] = 0$$

Since $G(z, z_k) \neq G(z, z_{k+1})$ then $\alpha_k = 0$ for $k = 1, 2, \dots, N$. Hence an \mathcal{L}^2 solution does not exist for this problem. However if we impose the additional constraints $I(\pm \frac{L}{2}) = 0$ then $D \cos kz + F \sin kz + Y$ may become an element in the range of the operator.

Whether in fact a forcing function is in the range of the operator is difficult to verify both theoretically and numerically. If a forcing function is not in the range of the operator, theoretically we should obtain a solution which diverges in an asymptotic sense. However this postulate may be difficult to verify numerically for certain problems.

As an example consider the partial sum of the series

$$S_N = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{N}$$

The partial sum S_N diverges as $N \rightarrow \infty$. This is because if we look at the following M terms of the series we find

$$\frac{1}{M+1} + \frac{1}{M+2} + \frac{1}{M+3} + \dots + \frac{1}{2M} > \frac{1}{2M} > \frac{1}{2M} + \frac{1}{2M} + \dots + \frac{1}{2M} > = \frac{1}{2}$$

Hence

$$S_\infty > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots = \infty$$

However if we program the series on the computer and ask the computer to

give us a result when the addition of the $N+1$ term does not change the partial sum by 10^{-10} (say) we would get a convergent result!

In conclusion, we must try to learn theoretically, as much about the problem as possible. Numerical methods may be applied as a last resort as it may be the only way to obtain a solution easily. The convergence of the numerically computed results is determined to a large extent by the theoretical analysis of the problem rather than apparent convergences in numerical computations.

6. CONCLUSION

In summary, we have brought out the following features.

- 1) The thin wire E-field Pocklington integral operator is unbounded whereas the Hallen E-field operator is bounded.
- 2) The inverse operator for Hallen E-field integral equation is unbounded.
- 3) A unified theory for the various iterative methods showing how the Fredholm equation of the first kind has been converted to a Fredholm equation of the second kind is presented.
- 4) The conditions under which the iterative methods converge both for the L^2 norm and the Chebyshev norm has been presented.
- 5) The monotonic rate of convergence of the sequence of solutions associated with iterative methods have been established for certain values of τ and for $Q \in R(H)$.
- 6) The numerical stability in the solution of the matrix equations for Galerkin's method for various expansion functions is examined and
- 7) The sequence of solutions I_N forms an asymptotic series for both the iterative and Galerkin's method when $Q \notin R(H)$.

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8. APPENDIX :

Theorem 2: For all $Q \in \text{Range of } H$, the sequence I_n generated by the recursion

$$I_{n+1} = [U - \tau H] I_n - \tau Q \triangle B I_n + Q' \quad (A.1)$$

Where U is the identity matrix and $||\tau H|| < 2$ with the initial guess $I_0 = Q'$ converges to I_e (the exact solution, if it exists) in the norm, i.e.

$\lim_{n \rightarrow \infty} ||I_n - I_e|| \rightarrow 0$ and the convergence is strictly monotone increasing, i.e., $I_k \uparrow I_e$.

Proof: The iterative process (A.1) converges as long as the norm of B is less than one. It is clear that if $|\tau| \cdot ||H|| < 2$ then $||B|| = ||U - \tau H|| < 1$. Now we have

$$I_e - I_{n+1} = I_e - B I_n - Q' = B[I_e - I_n]$$

By taking the norm of both sides and simplifying

$$||I_e - I_{n+1}|| \leq ||B|| \cdot ||I_e - I_n|| \leq \{||B||\}^{n+1} \cdot ||I_e - I_0||$$

Since $||B|| < 1$, as $n \rightarrow \infty$ we have

$$\lim_{n \rightarrow \infty} ||I_e - I_{n+1}|| = 0$$

and thus I_{n+1} converges to the exact solution.

That $I_k \uparrow I_e$ is seen easily as

$$\epsilon_{n+1} \triangleq I_e - I_{n+1}$$

and

$$\epsilon_n \triangleq I_e - I_n$$

are related by

$$\epsilon_{n+1} = B \epsilon_n$$

and so

$$||\epsilon_{n+1}|| \leq ||B|| \cdot ||\epsilon_n|| \leq ||\epsilon_n||$$

and with equality if and only if $\epsilon_n = 0$. It follows that if n_0 is the smallest integer for which $||\epsilon_{n+1}|| = ||\epsilon_n||$ then

$$\epsilon_n = 0, \text{ for } n \geq n_0 \text{ and } ||\epsilon_{n+1}|| < ||\epsilon_n|| \text{ for } n < n_0, \text{ i.e.}$$

$I_n \uparrow I_e$ and theorem 2 is proved.

Theorem 3: Consider the operator equation $HI = Q$ in a finite dimensional space N . Let the unknown I be expanded in terms of the normalized basis functions ψ_i . Define $\text{cond}[\hat{H}] = \frac{\lambda_{\max}[\hat{H}]}{\lambda_{\min}[\hat{H}]}$ in the given N dimensional space.

Let $\text{cond}[G]$ and $\text{cond}[E]$ be the condition numbers of the Galerkin matrix $[G_{ij} = \langle H\psi_i, \psi_j \rangle]$ and of the Gram matrix $[E_{ij} = \langle \psi_i, \psi_j \rangle]$, respectively. Then

$$\text{cond}[G] \leq \text{cond}[\hat{H}] \cdot \text{cond}[E]$$

Proof: Let $I = \sum_{i=1}^N \alpha_i \psi_i$, then from [2, p. 341]

$$\begin{aligned} |\langle HI, I \rangle| &= \left| \sum_{i=1}^N \sum_{j=1}^N \alpha_i \bar{\alpha}_j \langle H\psi_i, \psi_j \rangle \right| \leq ||\hat{H}|| \cdot \left| \sum_{i=1}^N \alpha_i \psi_i \right|^2 = ||\hat{H}|| \cdot \langle EI, I \rangle \\ &\leq ||\hat{H}|| \cdot \lambda_{\max}[E] \cdot ||\alpha||^2 \end{aligned}$$

since $\langle I, I \rangle = ||\alpha||^2$ we have

$$\frac{|\langle HI, I \rangle|}{\langle I, I \rangle} \leq ||\hat{H}|| \cdot \lambda_{\max}[E]$$

from which it follows that

$$\lambda_{\max}[G] \leq \|\hat{H}\| \cdot \lambda_{\max}[E]$$

Also since

$$|\langle HI, I \rangle| \geq \frac{\left\| \sum_{i=1}^N \alpha_i \psi_i \right\|}{\|\hat{H}^{-1}\|} = \frac{\langle EI, I \rangle}{\|\hat{H}^{-1}\|} \geq \frac{\lambda_{\min}[E] \cdot \|\alpha\|^2}{\|\hat{H}^{-1}\|}$$

so

$$\frac{|\langle HI, I \rangle|}{\langle I, I \rangle} \geq \frac{\lambda_{\min}[E]}{\|\hat{H}^{-1}\|}$$

from which it follows

$$\lambda_{\min}[G] \geq \frac{\lambda_{\min}[E]}{\|\hat{H}^{-1}\|}$$

Hence we have

$$\frac{\lambda_{\max}[G]}{\lambda_{\min}[G]} \leq \|\hat{H}\| \cdot \|\hat{H}^{-1}\| \cdot \frac{\lambda_{\max}[E]}{\lambda_{\min}[E]}$$

$$\text{cond}[G] \leq \text{cond}[\hat{H}] \cdot \text{cond}[E].$$

Theorem 4: If $Q \notin R(H)$ then the sequence of approximations I_n generated by

$$I_{n+1} = BI_n + Q$$

with the initialization $I_0 = Q$ yields the following relationships

$$1) \lim_{n \rightarrow \infty} \|R_{n+1} - R_n\| = 0, \text{ where } R_n = HI_n - Q$$

and

$$ii) \lim_{n \rightarrow \infty} \|I_n\| = \infty$$

Proof: i) We have

$$R_{n+1} - R_n = H[I_{n+1} - I_n]$$

and since the operator H is bounded [i.e. $\|H\| < M$] we have

$$\|R_{n+1} - R_n\| \leq M \|I_{n+1} - I_n\| \leq M \|\varepsilon_{n+1} - \varepsilon_n\|$$

$$\leq M \cdot \|B - U\| \cdot \|\varepsilon_n\| \leq M \cdot \|B - U\| \cdot \{\|B\|\}^n \|\varepsilon_0\|$$

Hence $\lim_{n \rightarrow \infty} \|R_{n+1} - R_n\| = 0$ as $\lim_{n \rightarrow \infty} \{\|B\|^n\} \rightarrow 0$.

ii) If $\lim_{n \rightarrow \infty} \|I_n\| = \infty$ does not hold, then we have $\lim_{n \rightarrow \infty} \|I_n\| < \infty$

(i.e a bounded sequence). Thus there is a subsequence I'_n which is bounded in norm. Now if we put the operator equation in a Hilbert space setting (now we can only talk about the \mathcal{L}^2 norm only) and since a Hilbertspace is weakly compact [2] one can always extract from I'_n another sequence I''_n which converges weakly to some element I of the Hilbert space, i.e. $I''_n \xrightarrow{w} I$. Also we have from i) $\lim_{n \rightarrow \infty} H I_n \xrightarrow{s} Q$ (strong convergence in norm). However as H is a bounded operator we have

$$\lim_{n \rightarrow \infty} H I''_n \xrightarrow{w} H I$$

and also

$$\lim_{n \rightarrow \infty} H I_n \xrightarrow{s} Q$$

Since the weak and strong limits of a sequence must coincide, $H I''_n = Q$.

This means $Q \in R(H)$, a contradiction.

Theorem 5: The sequence I_n as derived in ii) of Theorem 4 indeed forms an asymptotic series (i.e. for good accuracy the series has to be truncated after a finite number of terms otherwise the results may be worse).

Proof: To demonstrate the source of divergence in I_n we assume

$$HI_0 = Q_0 \text{ with } I_1 = Q_0 + \Delta Q$$

then

$$I_n = I_0 - [B]^n I_0 + \theta_{n-1} \text{ for } n \leq 1$$

where

$$\theta_{n-1} = \sum_{i=0}^{n-1} [B]^i \Delta Q$$

If $Q_0 + \Delta Q \notin R(H)$ then by theorem 4, $\lim_{n \rightarrow \infty} \|\theta_{n-1}\| = \infty$, since

$\lim_{n \rightarrow \infty} [B]^n I_0 = 0$. Observe that this holds irrespective of the size of

$|\Delta Q|$. Now the error in the iterates is obtained as

$$\epsilon_n = I_0 - I_n = [B]^n I_0 - \theta_{n-1}$$

Note that the norm of the first term is monotonically decreasing and thus it is evident that the algorithm should be terminated after a certain optimum number of steps. Unfortunately the exact number of iterations depends on the particular Q under consideration and the growth rate of $\|\theta_{n-1}\|$ versus the decay rate of $\|[B]^n I_0\|$.

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